

# NUMERICAL INVARIANTS OF FANO SCHEMES OF LINEAR SUBSPACES ON COMPLETE INTERSECTIONS

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**ABSTRACT.** The goal of this paper is to explore the genus and degree of the Fano scheme of linear subspaces on a complete intersection in a complex projective space. Firstly, suppose that the expected dimension of the Fano scheme is one, we prove a genus-degree formula. Secondly, we give a degree formula for the Fano scheme.

## 1. INTRODUCTION

Let  $X$  be a general complete intersection of type  $\underline{d} = (d_1, \dots, d_r)$  in the projective space  $\mathbb{P}^n$  over the complex field  $\mathbb{C}$ , provided that  $n, d_1, \dots, d_r$  are natural numbers with  $n \geq 4, d_i \geq 2$  for all  $i$ . Recall that the Fano scheme  $F_k(X)$  parametrizing linear subspaces of dimension  $k$  contained in  $X$  is a smooth subscheme of the Grassmannian  $G(k+1, n+1)$  of linear subspaces of dimension  $k$  in  $\mathbb{P}^n$ , provided that

$$(k+1)(n-k) \geq \sum_{i=1}^r \binom{d_i+k}{k}$$

and  $X$  is not a quadric, in which last case  $n \geq 2k+r$  is required (see [4, Corollary 2.2] or [6, Theorem 2.1]). For the basic properties of  $F_k(X)$ , we refer to [1, 2, 4, 6, 14]. For the important applications of  $F_k(X)$  to the geometry of  $X$ , we refer to [3, 5, 9]. Some numerical invariants of  $F_k(X)$  have been studied by many authors. For instance, the Picard number of  $F_k(X)$  was shown in [6, Proposition 3.1] and [13, Theorem 0.3]. The degree of  $F_k(X)$  under the Plücker embedding were formulated in [6, Theorem 4.3] and [11, Theorem 1.1]. In this paper, we explore the numerical invariants of  $F_k(X)$ . For convenience, we set

$$\delta(n, \underline{d}, k) = (k+1)(n-k) - \sum_{i=1}^r \binom{d_i+k}{k},$$

which is the expected dimension of  $F_k(X)$ . Our main results are the following:

**Theorem 1.** *With the notations as above, if  $\delta(n, \underline{d}, k) = 1$ , then the Fano scheme  $F_k(X)$  is a connected smooth projective curve of degree  $d$  and genus  $g$  satisfying the*

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following formula:

$$g = 1 + \frac{\sum_{i=1}^r \binom{d_i + k}{k + 1} - n - 1}{2} d.$$

**Theorem 2.** *With the notations as above, if  $\delta(n, \underline{d}, k) \geq 0$ , then the degree  $d$  of  $F_k(X)$  under the Plücker embedding is given by*

$$d = \frac{c(n, \underline{d}, k)}{(k + 1)!},$$

where  $c(n, \underline{d}, k)$  is the coefficient of  $x_0^n \cdots x_k^n$  in the polynomial

$$\prod_{i=1}^r \prod_{a_0 + \cdots + a_k = d_i, a_i \in \mathbb{N}} (a_0 x_0 + \cdots + a_k x_k) (x_0 + \cdots + x_k)^{\delta(n, \underline{d}, k)} \prod_{i \neq j} (x_i - x_j).$$

The statement of Theorem 1 can be viewed as a natural generalization of the work of Markusevich [16] and Tennison [17]. Meanwhile, the statement of Theorem 2 seems to be similar with that of [6, Theorem 4.3]. However, here we consider the coefficient of the monomial  $x_0^n \cdots x_k^n$  in the product of the polynomial

$$\prod_{i=1}^r \prod_{a_0 + \cdots + a_k = d_i, a_i \in \mathbb{N}} (a_0 x_0 + \cdots + a_k x_k) (x_0 + \cdots + x_k)^{\delta(n, \underline{d}, k)}$$

by the discriminant

$$\Delta = \prod_{i \neq j} (x_i - x_j)$$

instead of that of the monomial  $x_0^n x_1^{n-1} \cdots x_k^{n-k}$  in the product of the same polynomial by the Vandermonde determinant

$$a_\delta = \prod_{i < j} (x_i - x_j).$$

In particular, the proof of Theorem 2 is completely different from that of [6, Theorem 4.3]. We apply an integral formula for Grassmannians which has been recently explored in [12, Theorem 3]. The rest of the paper is organized as follows: Section 2 present the proof of Theorem 1. Theorem 2 is proved in Section 3.

## 2. PROOF OF THEOREM 1

For the proof of Theorem 1, we first prove some lemmas.

**Lemma 1.** *If  $E$  is a vector bundle of rank  $k + 1$  and  $\text{Sym}^m E$  is the  $m$ -th symmetric power of  $E$ , then the rank of  $\text{Sym}^m E$  is  $\binom{m+k}{k}$  and*

$$c_1(\text{Sym}^m E) = \binom{m+k}{k+1} c_1(E),$$

where  $c_1(E)$  is the first Chern class of the vector bundle  $E$ .

*Proof.* We prove by induction on  $m$  and  $k$ . For  $m = 1$ , we have  $\text{Sym}^1 E = E$ , so the rank of  $\text{Sym}^1 E$  is  $k+1$  and  $c_1(\text{Sym}^1 E) = c_1(E)$  for all  $k$ , the conclusion is true. Assume that the conclusion holds for all  $m' \leq m$  and  $k' \leq k$ . By arguments similar to those in [8, Lemma 7.6], we consider an exact sequence of vector bundles

$$0 \longrightarrow L \longrightarrow E \longrightarrow E' \longrightarrow 0,$$

where  $L$  is a line bundle and  $E'$  is a vector bundle of rank  $k$ . Thus we have

$$c_1(E) = c_1(L) + c_1(E').$$

The universal property of the symmetric powers (see, for example, [7, Proposition A.2.2.d]) shows that for each  $m \geq 1$  there is an exact sequence

$$0 \longrightarrow L \otimes \text{Sym}^{m-1} E \longrightarrow \text{Sym}^m E \longrightarrow \text{Sym}^m E' \longrightarrow 0.$$

By induction, the rank of  $\text{Sym}^m E$  is

$$\binom{m+k-1}{k-1} + \binom{m-1+k}{k} = \binom{m+k}{k},$$

and

$$\begin{aligned} c_1(\text{Sym}^m E) &= c_1(L \otimes \text{Sym}^{m-1} E) + c_1(\text{Sym}^m E') \\ &= \binom{m+k-1}{k} c_1(L) + c_1(\text{Sym}^{m-1} E) + \binom{m+k-1}{k} c_1(E') \\ &= \binom{m+k-1}{k} c_1(E) + \binom{m+k-1}{k+1} c_1(E) \\ &= \binom{m+k}{k+1} c_1(E). \end{aligned}$$

□

**Lemma 2.** ([8, Proposition 5.25]) *The first Chern class of the tangent bundle  $T_G$  of the Grassmannian  $G = G(k+1, n+1)$  is*

$$c_1(T_G) = (n+1)\sigma_1,$$

where  $\sigma_1 = c_1(S^\vee) = c_1(Q)$  is a hyperplane section of  $G$  in the Plücker embedding,  $S$  and  $Q$  are respectively universal sub and quotient bundles on  $G$ , and  $S^\vee$  is the dual of  $S$ .

*Proof.* The tangent bundle  $T_G$  is expressed as the tensor product  $S^\vee \otimes Q$ . Thus we have

$$c_1(T_G) = c_1(S^\vee \otimes Q) = \text{rank}(Q)c_1(S^\vee) + \text{rank}(S^\vee)c_1(Q).$$

Since  $\text{rank}(Q) = n - k$ ,  $\text{rank}(S^\vee) = k + 1$ , and  $c_1(S^\vee) = c_1(Q) = \sigma_1$ , hence  $c_1(T_G) = (n+1)\sigma_1$  as desired. □

**Lemma 3.** ([8, Proposition 6.4]) *Let  $X \subset \mathbb{P}^n$  be a general complete intersection of type  $(d_1, \dots, d_r)$ . The scheme  $F = F_k(X)$  is the zero locus of a global section of the vector bundle*

$$\mathcal{F} = \bigoplus_{i=1}^r \text{Sym}^{d_i} S^\vee.$$

*Proof.* Assume that  $X$  is the intersection of  $r$  hypersurfaces  $X_1, \dots, X_r$  with  $\deg(X_i) = d_i$  for all  $i$ . Each  $F_k(X_i)$  is the zero locus of a global section  $s_i$  of  $\text{Sym}^{d_i} S^\vee$ . Thus the scheme  $F$ , which is the intersection of the  $F_k(X_i)$ , is the zero locus of a global section  $s = (s_1, \dots, s_r)$  of the vector bundle  $\mathcal{F}$ .  $\square$

*Proof of Theorem 1.* The dimension and smoothness of the Fano scheme  $F = F_k(X)$  are proved in [4, Corollary 2.2] and [6, Théorème 2.1], also see further discussion in [8, Chapter 6] when  $X$  is a hypersurface. The connectedness of  $F$  is proved in [4, Theorem 4.1], also see in [14, Theorem 0.1] for the hypersurface case.

Recall that the degree of a subvariety of a projective space is defined to be the number of its intersection points with a generic linear subspace of complementary dimension. The degree  $d$  of  $F$ , considered as a subvariety of a projective space thanks to Plücker embedding, is computed by the following formulas:

$$(1) \quad d = \int_G [F] \cdot \sigma_1^{\delta(n, \underline{d}, k)},$$

where  $\int_G \alpha$  denotes the degree of the 0-dimensional cycle class  $\alpha$  on  $G$  defined in [10, Definition 1.4]. If the expected dimension  $\delta(n, \underline{d}, k) = 1$ , then the degree  $d$  and genus  $g$  of  $F$  is computed as follows:

$$d = \int_G [F] \cdot \sigma_1$$

and

$$g = 1 - \chi(\mathcal{O}_F) = 1 - \frac{1}{2} \int_F c_1(T_F),$$

where  $T_F$  is the tangent bundle of  $F$ . In order to determine  $c_1(T_F)$ , we consider the normal bundle sequence

$$0 \longrightarrow T_F \longrightarrow T_G|_F \longrightarrow N_{F/G} \longrightarrow 0,$$

where  $N_{F/G}$  is the normal bundle of  $F$  in  $G$ . By Lemma 3,  $F$  is the zero locus of a section of  $\mathcal{F}$ , so  $N_{F/G}$  is isomorphic to  $\mathcal{F}|_F$ . By Lemma 1 and Lemma 2, we have

$$\begin{aligned}
\int_F c_1(T_F) &= \int_F (c_1(T_G|_F) - c_1(\mathcal{F}|_F)) \\
&= \int_G (c_1(T_G) - c_1(\mathcal{F})) \cdot [F] \\
&= \int_G \left( (n+1)\sigma_1 - \sum_{i=1}^r c_1(\text{Sym}^{d_i} S^\vee) \right) \cdot [F] \\
&= \int_G \left( (n+1) - \sum_{i=1}^r \binom{d_i+k}{k+1} \right) \sigma_1 \cdot [F] \\
&= \left( n+1 - \sum_{i=1}^r \binom{d_i+k}{k+1} \right) \int_G \sigma_1 \cdot [F] \\
&= \left( n+1 - \sum_{i=1}^r \binom{d_i+k}{k+1} \right) d.
\end{aligned}$$

In summary, the genus-degree formula is obtained as desired.  $\square$

**Example 1.** [17, Section 2] Let  $X \subset \mathbb{P}^4$  be a general quartic threefold. In this case,  $\delta(4, (4), 1) = 1$  and  $F_1(X)$  is a smooth projective curve of degree  $d = 320$  and genus  $g = 801$  satisfying

$$g = 1 + \frac{\binom{5}{2} - 5}{2}d = 1 + \frac{5}{2}d.$$

More generally, if  $X \subset \mathbb{P}^n$  ( $n \geq 4$ ) be a general hypersurface of degree  $2n - 4$ , then  $F_1(X) \subset \mathbb{P}^{\binom{n+1}{2}-1}$  is a smooth projective curve of degree  $d$  and genus  $g$  satisfying the following formula:

$$g = 1 + \frac{\binom{2n-3}{2} - n - 1}{2}d.$$

**Example 2.** [16, Theorem 2.2 (i)] Let  $X \subset \mathbb{P}^5$  be a general complete intersection of type  $(2, 3)$ . In this case,  $\delta(5, (2, 3), 1) = 1$  and  $F_1(X)$  is a smooth projective curve of degree  $d = 180$  and genus  $g = 271$  satisfying

$$g = 1 + \frac{\binom{3}{2} + \binom{4}{2} - 6}{2}d = 1 + \frac{3}{2}d.$$

More generally, if  $X \subset \mathbb{P}^n$  ( $n \geq 5$ ) be a general complete intersection of type  $(n - 3, n - 2)$ , then  $F_1(X)$  is a smooth projective curve of degree  $d$  and genus  $g$  satisfying

the following formula:

$$g = 1 + \frac{\binom{n-2}{2} + \binom{n-1}{2} - n - 1}{2}d.$$

**Example 3.** [16, Theorem 2.2 (ii)] Let  $X \subset \mathbb{P}^6$  be a general complete intersection of type  $(2, 2, 2)$ . Then  $\delta(6, (2, 2, 2), 1) = 1$  and  $F_1(X)$  is a smooth projective curve of degree  $d = 128$  and genus  $g = 129$  satisfying

$$g = 1 + \frac{\binom{3}{2} + \binom{3}{2} + \binom{3}{2} - 7}{2}d = 1 + d.$$

More generally, if  $X \subset \mathbb{P}^n$  ( $n \geq 6$ ) be a general complete intersection of type  $(2, n-4, n-4)$ , then  $F_1(X)$  is a smooth projective curve of degree  $d$  and genus  $g$  satisfying the following formula:

$$g = 1 + \frac{2\binom{n-3}{2} - n + 2}{2}d.$$

### 3. PROOF OF THEOREM 2

By the Gauss-Bonnet formula (see, for example, [15, Subsection 3.5.3]), the class of  $F_k(X)$  is the top Chern class of the vector bundle  $\mathcal{F}$ . If  $\delta(n, \underline{d}, k) \geq 0$ , then the degree  $d$  of  $F_k(X)$  can be expressed as follows:

$$(2) \quad d = \int_{G(k+1, n+1)} \prod_{i=1}^r c_{\text{top}}(\text{Sym}^{d_i} S^\vee) \cdot c_1(S^\vee)^{\delta(n, \underline{d}, k)},$$

where  $c_{\text{top}}(E)$  is the top Chern class of the vector bundle  $E$ . By the splitting principle ([10, Remark 3.2.3 and Example 3.2.6]), the characteristic class

$$\prod_{i=1}^r c_{\text{top}}(\text{Sym}^{d_i} S^\vee) \cdot c_1(S^\vee)^{\delta(n, \underline{d}, k)}$$

is represented by the symmetric polynomial

$$(-1)^{(k+1)(n-k)} \prod_{i=1}^r \prod_{a_0 + \dots + a_k = d_i, a_i \in \mathbb{N}} (a_0 x_0 + \dots + a_k x_k) (x_0 + \dots + x_k)^{\delta(n, \underline{d}, k)}.$$

Note that  $x_0, \dots, x_n$  are the Chern roots of the tautological subnumdle  $S$  on the Grassmannian  $G(k+1, n+1)$ . By [12, Theorem 3], Theorem 2 follows.

As a corollary, we have the following result.

**Corollary 1.** *Suppose that  $\delta(n, \underline{d}, k) = 0$ . Then the number of linear subspaces of dimension  $k$  contained in a generic complete intersection of type  $\underline{d} = (d_1, \dots, d_r)$  in*

$\mathbb{P}^n$  is equal to  $\frac{c(n, \underline{d}, k)}{(k+1)!}$ , where  $c(n, \underline{d}, k)$  is the coefficient of the monomial  $x_0^n \cdots x_k^n$  in the polynomial

$$\prod_{i=1}^r \prod_{a_0 + \cdots + a_k = d_i, a_i \in \mathbb{N}} (a_0 x_0 + \cdots + a_k x_k) \prod_{i \neq j} (x_i - x_j).$$

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